

# → Sound and Acoustic Emission

## a) Review

Now, consider

- compressible potential flow

$$\underline{v} = \underline{\nabla} \phi, \quad \underline{\nabla} \cdot \underline{v} \neq 0$$

- adiabatic eqn. state

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = -\rho \underline{\nabla} \cdot \underline{v}$$

$$\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right) = -\underline{\nabla} p$$

linearizing:  $\rho = \rho_0 + \tilde{\rho}$   
 $\underline{v} = \underline{v}_0 + \tilde{\underline{v}}$

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \tilde{\underline{v}}$$

$$\rho_0 \frac{\partial \tilde{\underline{v}}}{\partial t} = -\underline{\nabla} \tilde{p}$$

$$\tilde{p} = \left( \frac{\partial p}{\partial \rho} \right)_s \tilde{\rho}$$

↓  
adiabatic

$$\underline{v} = \underline{\nabla} \phi \Rightarrow$$

$$\tilde{p} = c_s^2 \tilde{\rho}$$

$$\tilde{p} = -\rho_0 \frac{\partial \phi}{\partial t}$$

$$p = p_0 \left( \rho / \rho_0 \right)^\gamma$$

$$\therefore c_s^2 = \gamma p / \rho$$

relates  $\phi$ ,  $p$ .

$$-c\omega \tilde{\rho} = \rho_0 k^{\rightarrow} \tilde{\phi}$$

$$+ i\omega \rho_0 \tilde{\phi} = c_s^2 \tilde{\rho}$$

$$\Rightarrow \omega^2 = k^{\rightarrow} c_s^2$$

↓  
sound / acoustic wave.

or  $\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla^2 \tilde{\phi}$

$$- \frac{\partial \tilde{\phi}}{\partial t} = c_s^{\rightarrow} \tilde{\rho}$$

$$\Rightarrow \nabla^2 \tilde{\phi} - \frac{1}{c_s^2} \frac{\partial^2 \tilde{\phi}}{\partial t^2} = 0$$

wave eqn.

Note: - sound is longitudinal wave  $\Leftrightarrow$

contrast string

-  $c_s$  is sound speed.

$$- \underline{k} = \frac{\omega}{c_s} \hat{n}$$

$c_s$  ↓

unit vector in direction propagation

Also, recall energy / Poynting thm for acoustic waves, i.e.

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \underline{S} = 0, \quad \text{in absence of sources / dissipation}$$

$\underline{\Sigma} \equiv$  energy density

$$\underline{\Sigma} = \underbrace{\frac{1}{2} \rho_0 \tilde{v}^2}_{\text{kinetic (mechanical)}} + \underbrace{\frac{1}{2} c_s^2 \rho_0 \left(\frac{\tilde{\rho}}{\rho_0}\right)^2}_{\text{potential (internal)}}$$

$\underline{S} \equiv$  wave energy density flux

$$\underline{S} = c_s^2 \tilde{\rho} \tilde{\underline{v}}$$

$$\tilde{\underline{v}} = \tilde{v} \hat{n} = c \underline{k} \tilde{\phi}$$

$$\rho_0 + i\omega \tilde{\rho} = c_s^2 \tilde{\rho}$$

$$\begin{aligned} \therefore \underline{S} &= c_s^2 \frac{i\omega \tilde{\rho}}{c_s^2} \times c \underline{k} \tilde{\phi} \rho_0 \\ &= \rho_0 k^2 |\tilde{\phi}|^2 c_s \hat{n} \end{aligned}$$

$$\underline{S} = c_s \underline{\Sigma} \hat{n}$$

$$\text{n.b.} \left\{ \begin{array}{l} \underline{S} = \underline{v}_g \underline{\Sigma}, \\ \text{or general.} \end{array} \right.$$

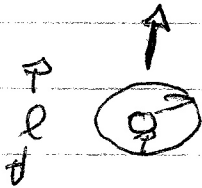
n.b. in electron theory,  $\hat{n} = \frac{\nabla \psi}{|\nabla \psi|}$

$$\phi = \phi_0 e^{i\psi}, \text{ etc.}$$

## b.) Acoustic Radiation

Radiation  $\rightarrow$  How is sound generated?

i.e.



i.e. relate far-field  
 $r \gg \ell, \lambda$   
 intensity, etc. to  
source structure  
 and dynamics.

$\infty$ , first

## c.) Spherical Waves.

- when sound field depends only on distance from some point  $\Rightarrow$  spherical symmetry
- sound wave  $\rightarrow$  spherical wave

i.e. if  $\phi = \phi(r, t)$ , then

$$\frac{\partial^2 \phi}{\partial t^2} = c_s^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right)$$

suggests immediately  $\phi = \frac{f(r, t)}{r}$

$$\frac{\partial^2 \phi}{\partial t^2} = c_s^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left( \frac{-f}{r^2} + \frac{f'}{r} \right) \right\}$$

$$= c_s^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ -f + r f' \right\}$$

$$\frac{1}{r} \frac{\partial^2 f}{\partial t^2} = c_s^2 \frac{1}{r^2} \left\{ -\cancel{f'} + \cancel{f'} + r f'' \right\}$$

$$\frac{\partial^2 f}{\partial t^2} = c_s^2 \frac{\partial^2 f}{\partial r^2}$$

$$f = f_1(ct - r) + f_2(ct + r)$$

100

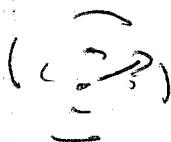
$$\phi = f_1 \frac{(ct - r)}{r} + f_2 \frac{(ct + r)}{r}$$

general spherical wave soln.

outgoing

(causal)

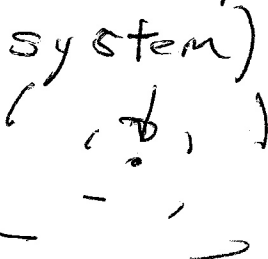
relevant to radiation



incoming

(almost always non-causal)

→ exception: bounded system



For spherical wave with  $kr \gg 1$ ;  
(far field radiation)

$$\underline{v} = v \hat{r} = -\frac{f'}{c}$$

$$f' = -c \dot{f}$$

$$\underline{E} = \rho_0 \frac{f'^2}{r^2}, \quad \underline{S} = \hat{r} c \rho_0 \frac{f'^2}{r^2}$$

and  $r^2 \underline{S} = \hat{r} c \rho_0 f'^2$

- intensity  $\sim 1/r^2$  so flux thru surface at  $r \sim \text{const.}$

- derivative terms }  $\rightarrow$  flux decreases in far field

- if no source,  $\phi$  must be finite at  $r=0$

$$\phi = \frac{f_1(ct-r)}{r} + \frac{f_2(ct+r)}{r}$$

$$= \frac{f(ct-r)}{r} - \frac{f(ct+r)}{r}$$

- outgoing, monochromatic wave

$$\phi = \frac{A}{r} e^{i(kr - \omega t)}$$

- standing monochromatic wave

$$\phi = A \frac{\sin kr}{r} e^{-i\omega t}$$

### (ii) Sound Emission - I

- am concerned with acoustic emission from moving body

↳ deformation → compression  
[displacement]

now, acoustics ↔ potential flow:

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$$\nabla^2 \phi = + \frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} \rightarrow \text{eqn.}$$

$$u = \frac{\partial \phi}{\partial n} \rightarrow \text{b.c.}$$

↓  
body velocity      ↳ fluid motion normal to body

- in all cases :  $|v| \ll c_s$

$$|v| \sim \omega a \Rightarrow \omega a \ll c_s$$

$\downarrow$   
 displacement

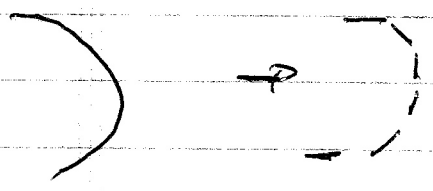
$a \ll \lambda$   
 $\downarrow$   
 wavelength.

- can consider 2 cases :

- 1)  $\lambda \ll l$  ~~body size~~ (high freq./short wavelength)
- 2)  $\lambda \gg l$  (low freq./long wavelength)

1)  $\lambda \ll l$  (easy)

- treat body as array of planar facets, each  $l' > \lambda$



body motion/pulsation  $\rightarrow$  each facet plate moves, emits sound.



So, each facet emits plane wave. Thus,  
for facet

$$\underline{S} = c_0 \rho V_n^2 \hat{n} = c_0 \rho u_n^2 \hat{n}$$

$$I = P_{\text{rad}}$$

$\therefore I = \int d\mathbf{a} \cdot \underline{S} \rightarrow$  emitted intensity  
radiated power (Watts)

$$I = \int d\mathbf{a} \hat{n} \cdot \underline{S} = \int d\mathbf{a} c_0 \rho V_n^2$$

$$I = \int d\mathbf{a} c_0 \rho \underbrace{V_n^2}_{u_n^2} \quad (\text{Radiated Power})$$

n.b. for  $I$ ;

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \underline{S} = 0$$

$$\int d^3x \frac{\partial \mathcal{E}}{\partial t} + \int d^3x \nabla \cdot \underline{S} = 0$$

$$\frac{\partial \mathcal{E}}{\partial t} = - \int d^3x \nabla \cdot \underline{S} = + \int d\mathbf{a} \cdot \underline{S}$$

$\hookrightarrow I$  within

Now,

2)  $l \ll \lambda$  (non-trivial)

for emission at frequency  $\omega$  :

$$\nabla^2 \phi + \frac{\omega^2}{c_s^2} \phi = 0$$

$$\left\{ \begin{array}{l} \sim 1/\lambda^2 \end{array} \right.$$

$1/l^2$  (near body)

so, can break prob into 3 zones

i) ultra-near field

$$r \sim l \ll \lambda$$

$$\nabla^2 \sim 1/l^2 \gg \omega^2/c_s^2$$

$$\rightarrow \nabla^2 \phi = 0$$

(potential flow)

and

body configuration details matter.

ii) near field

$$\lambda > r > l$$

$$\nabla^2 \phi = 0$$

but, represent by multipole expansion!

(ii.) Far field  $\rightarrow$  region sound radiated to...

$$r \gg \lambda \quad \phi = - \frac{f(t - r/c_s)}{r}$$

$\downarrow$   
outgoing  
spherical  
wave.

$$v_r = \frac{1}{c} \frac{f'(t - r/c_s)}{r}$$

$\rightarrow$  idea will be to match (ii.) to (i.)!

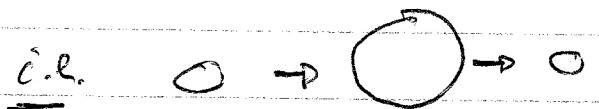
Now, in general:  $\nabla^2 \phi = 0$

$$\Rightarrow \phi = \frac{-Q}{r} + \frac{A \cdot \nabla}{r} (1/r) + \dots$$

$\downarrow$   
dipole moment

consider term-by-term, ...

(i) monopole



monopole  $\rightarrow$  body pulsates  
 $\Leftrightarrow$  volume varies

Aside: Multipole Expansions

$$\nabla^2 \phi = -4\pi \rho$$

$\rho \equiv$  charge on arbitrary source

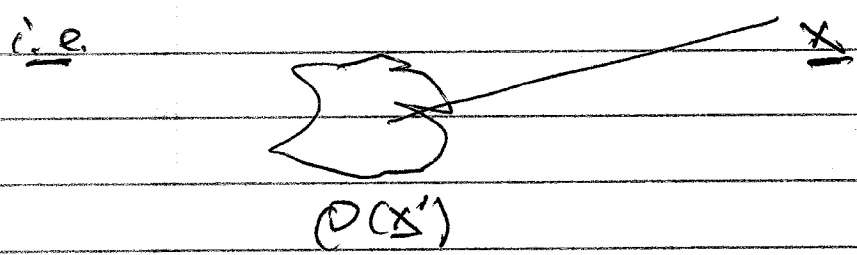
i.e.   
 Poisson Eqn.   
 or   
 $\nabla \cdot \underline{V} = \rho, \underline{V} = -\nabla \phi$

$\Rightarrow$

$$\phi(x) = \int \frac{d^3x' \rho(x')}{|x-x'|}$$

$$G(x, x') = \frac{1}{4\pi |x-x'|}$$

Consider localized source/charge distribution:



taking source  $\sim$  origin,  $|x'| \ll |x|$   
 so expand

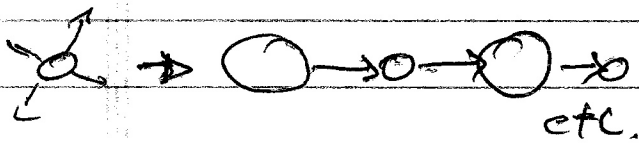
$$\phi = \int d^3x' \rho(x') \left[ \frac{1}{|x|} - \frac{x' \cdot \nabla}{|x|} + \frac{1}{2} \frac{x'_\alpha x'_\beta}{|x|^3} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{|x|} \right]$$

$\textcircled{1}$   $\textcircled{2}$   $\textcircled{3}$   
 $= \frac{Q}{|x|} - \left( \frac{d}{A} \right) \cdot \nabla \frac{1}{|x|} + \frac{1}{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{|x|}$   
 $\downarrow$   
 + ... acoustics

①  $\rightarrow$  monopole  $\rightarrow$  scalar  $\sim 1/r$

$$Q = \int d^3x' \rho(x')$$

Physics  $\left\{ \begin{array}{l} \text{e.s. total net charge} \\ \text{acoustics: total } \vec{v} \rightarrow \text{compression} \end{array} \right.$



$\rightarrow$  net, bulk property of source.

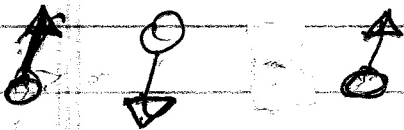
②  $\rightarrow$  dipole  $\rightarrow$  vector  $\sim 1/r^2$

$$\left( \frac{d}{A} \right) = \int d^3x' \rho(x') \mathbf{x}'$$

relative displacement

Physics  $\left\{ \begin{array}{l} \text{e.s. net charge polarization} \\ \text{acoustics: net displacement of body.} \end{array} \right.$

cell: sphere  $A(t) = \frac{R^3}{2} \underline{u}(t)$



vector defining dipole moment

(b)  $\rightarrow$  quadrupole  $\rightarrow$  tensor  $\rightarrow$  dyad

$$\sim 1/|x|^3$$

N.B. = for <sup>mass</sup> e.g.,

$$D_{\alpha\beta} = \int d^3x' \rho(x') \frac{x'_\alpha x'_\beta}{2}$$

but as  $\nabla^2 \frac{1}{|x|} = \delta_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left( \frac{1}{|x|} \right) = 0$


$$|x| \neq 0$$

$$D_{\alpha\beta} = \int d^3x' \frac{\rho(x')}{2} \left[ x'_\alpha x'_\beta - \frac{1}{3} r'^2 \delta_{\alpha\beta} \right]$$

" quadrupole moment of mass distribution is moment of inertia tensor.

- can view quadrupole as pair of opposite dipoles (just as dipole as pair of opposite charges)

c.e.


  
 longitudinal quadrupole      lateral quadrupole.

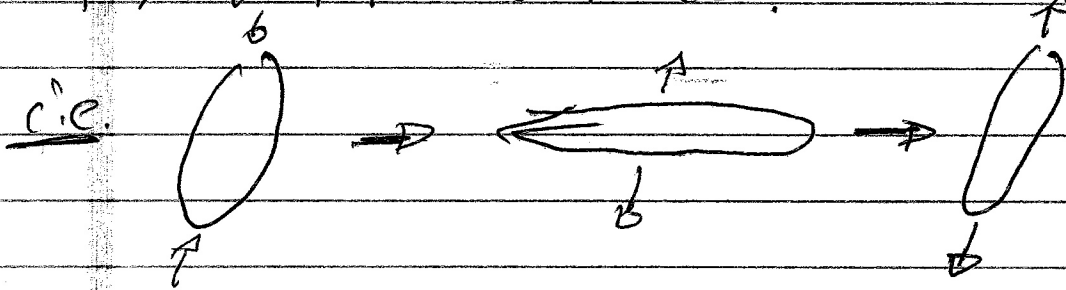
Physics:

(moments)

→ oscillation in distribution (shape) of  
e.s. charge at fixed total charge.

→ oscillation in (distribution) shape of  
acoustics.

body (which sets b.c. for fluid) at  
fixed total volume.



(but volume fixed!).

$$\phi = -a/r$$

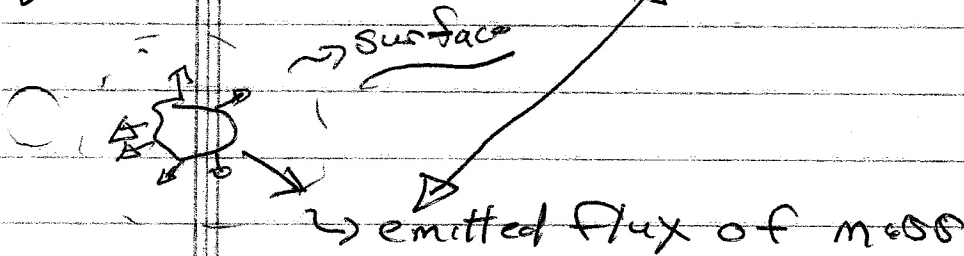
$$a \sim L^3/T$$

(dimensionally)

$$V_r = a/r^2$$

So for emitted mass flux due pulsations  
at some enclosing surface

$$\int 4\pi r^2 d\Omega \rho V_r = 4\pi \rho a$$



but then  $\frac{4\pi \rho a}{\rho} = \dot{V}$  mass flux

Rate of change in  
volume of pulsating  
object.

(N.B.  $\nabla^2 \phi = 0$   
in near field  
 $\Rightarrow \odot \nabla \cdot \underline{v} = 0$ )

$$\phi = \frac{-\dot{V}}{4\pi r} = \frac{-\dot{V}(t - r/c_s)}{4\pi r}$$



So  $V_r = \frac{\partial \phi}{\partial r}$

$$= -\frac{1}{c_s} \frac{\partial \phi}{\partial t}$$

radial velocity of  $r$  set by volume change at retarded time  $t - r/c_s$ .

$$= \frac{1}{c_s} \frac{\ddot{V}}{4\pi r} = \frac{1}{c_s} \frac{\ddot{V}(t - r/c_s)}{4\pi r}$$

$$I = \int \hat{r} \cdot d\mathbf{a} \rho v_r^2 c_s$$

$$= \rho c_s \int da \frac{1}{c_s^2} \frac{|\dot{V}|^2}{(4\pi r)^2}$$

$$I = \frac{\rho}{(4\pi)^2 c_s} \int da \frac{|\ddot{V}|^2}{r^2}$$

time average  
→ radiated power

see 76a.

N.B.

- harmonic pulsations of  $\omega$

$$\ddot{V} \sim \omega \dot{V} \Rightarrow I \sim \omega^2, \text{ for given surface velocity amplitude}$$

- harmonic pulsations at fixed <sup>spatial</sup> amplitude given

For spherical surface of integration:

$$A = 4\pi r^2$$

$$\Rightarrow I = P_{\text{rad}} = \frac{p}{4\pi r^2} \overline{|\dot{V}|^2} \quad \rightarrow \text{total intensity of emitted sound.}$$

Example: L+L, Pg. 286 #3.

Sound from sphere executing small harmonic pulsations with

$$u_s = u_0 e^{-i\omega t} \hat{r}$$

Assume sphere has radius  $R$ , with  $\Delta R/R \ll 1$ .  
 $P_{\text{rad}}?$

Consider:

- 1) Low Frequency limit
- 2) High Frequency limit
- 3) unified

1) Can plug in, with

$$V = \frac{4}{3}\pi R^3$$

$$\dot{V} = 4\pi R^2 \dot{R}$$

$$\ddot{V} = 4\pi R^2 \ddot{R} + 8\pi R \dot{R}^2$$

~~$\Delta R/R$~~

$$\dot{R} = U_0 e^{-i\omega t}$$

$$\ddot{R} = -i\omega U_0 e^{-i\omega t}$$

so

$$\begin{aligned} \vec{V} &= -4\pi i\omega R^2 U_0 e^{-i\omega t_{ret}} \\ &= -4\pi i\omega R^2 U_0 e^{-i\omega(t - \frac{r}{c})} \\ &= 4\pi i\omega R^2 U_0 e^{i(kr - \omega t)} \end{aligned}$$

end

$$P_{rad} = \frac{\rho}{4\pi c^3} \overline{\vec{V}^2}$$

$$= \frac{\rho}{4\pi c^3} (4\pi)^2 \omega^2 R^4 \frac{U_0^2}{2} \rightarrow \text{time avg}$$

$$P_{rad} = \frac{2\pi \rho \omega^2 R^4 U_0^2}{c^3} \sim \omega^2 U_0^2$$

2) Can plug in, with

$$P_{rad} = c_0 \int ds \overline{u_n^2}$$

$$u_n = U_0 e^{-i\omega t}$$

$$\overline{u_n^2} = U_0^2 / 2$$

$$A = 4\pi R^2$$

$$\Rightarrow P_{\text{rad}} = c \epsilon_0 \frac{4\pi R^2 U_0^2}{2}$$

$$= 2\pi c \epsilon_0 R^2 U_0^2$$

3) Unified.

Now, - near field

$$\phi \approx \frac{q}{r} = \frac{qU_0}{r}$$

- far field

→ shift due to finite radius

$$\phi \sim \frac{e^{ik(r-R)}}{r} qU_0$$

$$\therefore \phi = \frac{qU_0}{r} e^{ik(r-R)}$$

$a \frac{\partial}{\partial r}$

$$\left. \frac{\partial \phi}{\partial r} \right|_R = U_0$$

$$\Rightarrow -\frac{qU_0}{R^2} + \frac{ik qU_0}{R} = U_0$$

$$\Rightarrow q = R^2 / (\epsilon k R - 1)$$

$$\therefore I = 2\pi R C_0 |U_0|^2 k^2 R^4 / (1 + k^2 R^2)$$

limits check ✓

$$\therefore I \sim \omega^4$$

ii.) dipole  $\rightarrow$  body moves  
c.e.  $\uparrow$

emission from oscillating sphere,

for pot. flow,

$$\phi = \underline{\nabla} \cdot \left[ \frac{\underline{A}(t)}{r} \right] = \underline{A}(t) \cdot \underline{\nabla} (1/r) \rightarrow \text{dipole potential}$$

$\therefore$  for  $r > l$

$$\phi = \underline{\nabla} \cdot \left[ \frac{\underline{A}(t - \frac{r}{c_s})}{r} \right]$$

and for  $r > \lambda$ , thus:

$$\phi = \frac{-\dot{\underline{A}}(t - r/c_s) \cdot \underline{n}^T}{c_s r}$$

$\underline{n}^T \equiv$  direction  
prop

but  $\underline{v} = \underline{\nabla} \phi$

$$\Rightarrow \underline{v} = \frac{-\ddot{\underline{A}} \cdot \underline{n}^T}{c_s^2 r} \underline{\nabla} \left( t - \frac{r}{c_s} \right)$$

so 
$$\underline{\nabla} \left( t - \frac{r}{c_s} \right) = -\frac{\underline{\hat{n}}}{c_s} = \frac{-\underline{\hat{n}}}{c_s} \nabla r$$

$$\underline{v} = \underline{\hat{n}} \left( \underline{\hat{n}} \cdot \underline{\ddot{A}} \left( t - \frac{r}{c_s} \right) \right) / c_s^2 r$$

$$I = \frac{\rho}{c_s^3} \int da \frac{\left( \underline{\hat{n}} \cdot \underline{\ddot{A}} \right)^2}{r^2}$$

time-avgd  
radiated  
power

- for given velocity amplitude ( $A \sim u$ )

$$I \sim \omega^4$$

- for given position amplitude

$$I \sim \omega^6$$

Integrating:

$$I = \frac{4\pi\rho}{3c_s^3} |\underline{\ddot{A}}|^2$$

Note: For (electric) dipole EM radiation, have

$$P_{\text{rad}} = \frac{2}{3c^3} |\ddot{d}|^2$$

→ Larmor  
Formula.

Analogy obvious.



Ex.  $P_{rad}$  from sphere oscillating at  $\omega$

$\uparrow$   
 $\circ$   $\downarrow$  with  $dx < R$   
 $\underline{u} = u_0 e^{-i\omega t}$

a) Low Frequency

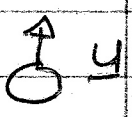
Flow is dipolar! Radiation is dipolar!

Acoustic Larmor Formula  $\Rightarrow$

$$P_{rad} = \frac{4\pi\rho}{3c^3} \overline{\dot{\underline{A}}^2}$$

For  $\underline{A}$ ;  $\nabla^2 \phi = 0$

$$\phi = \frac{-\alpha}{r} + \underline{A} \cdot \underline{\nabla} (1/r) + \dots$$



$$\approx \underline{A} \cdot \underline{\nabla} (1/r)$$

$$= \alpha \underline{u} \cdot \underline{\nabla} (1/r) = -\alpha u \cos\theta / r^2$$

$$V_n = \frac{2\alpha u \cos\theta}{r^3}$$

$$V_n \Big|_R = u \cos\theta \Rightarrow \alpha = \frac{1}{2} R^3$$

$$A = (R^3/2) \underline{u}$$

$$\Rightarrow P_{\text{rad}} = \frac{4\pi\rho}{3c_s^3} \frac{R^6}{4} \underline{\dot{u}}^2$$

$$\underline{u} = \underline{u}_0 e^{-i\omega(t-r/c_s)}$$

$$\Rightarrow P_{\text{rad}} = \frac{\pi\rho}{6c_s^3} R^6 \omega^4 |\underline{u}_0|^2$$

b) high frequency

$$P_{\text{rad}} = c_s \rho \int \underline{u}_{,n}^2 dS$$

$$= c_s \rho \left( \frac{4\pi R^2}{3} \right) \frac{|\underline{u}_0|^2}{2}$$

$$P_{\text{rad}} = \frac{2\pi\rho}{3} c_s |\underline{u}_0|^2$$

c) General:

$$\nabla^2 \phi + \frac{\omega^2}{c_s^2} \phi = \nabla^2 \phi + k^2 \phi = 0$$

$$\phi = -\frac{q}{r} + \underline{A} \cdot \underline{\nabla} (1/r) \quad \text{, } \underline{A} \sim \alpha \underline{u}$$

in near field,

$\therefore$  can write:  $\phi = \underline{u} \cdot \underline{\nabla} f(r)$

$$\Rightarrow \underline{u} \cdot \underline{\nabla} (\nabla^2 f + k^2 f) = 0$$

$$\therefore \nabla^2 f + k^2 f = \text{const.}$$

$$\Rightarrow f = \frac{A}{r} e^{i k (r - R)} + \text{const}$$

Matching  $\Rightarrow$

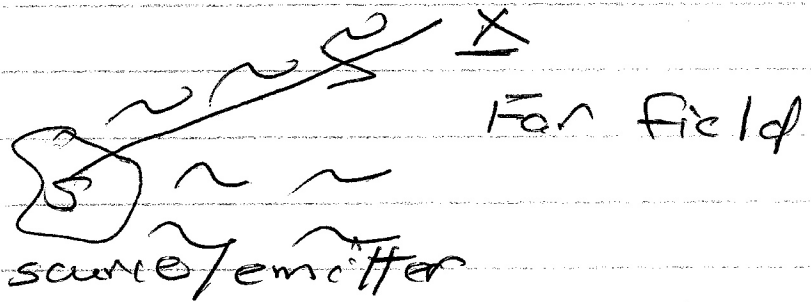
$$A = \frac{+u R^3}{2 - 2i k R - 4k^2 R^2} \quad \checkmark$$

## → General Theory of Acoustic Radiation

### i) Green's Function for Wave Equation

Now,

- Radiation Theory :



- Green's Function :  $\phi(\underline{x}, t) = \int d^3x' \int dt' \Theta(\underline{x} - \underline{x}', t - t') S(\underline{x}', t')$

↳ Response Function

∴ Wave equation Green's Function central to Radiation theory.

Obtain via :

- substitution
- contour integration

a) Substitution

$$c = c_s$$

$$\left( \nabla^2 - \frac{1}{c^2} \partial_t^2 \right) G(\underline{x} - \underline{x}', t - t') = -\delta^3(\underline{x} - \underline{x}') \delta(t - t')$$

$$G(\underline{x} - \underline{x}', \omega) = \int dt e^{i\omega(t-t')} G(\underline{x} - \underline{x}', t - t')$$

$$\left( \nabla^2 + \frac{\omega^2}{c^2} \right) G(\underline{x} - \underline{x}', \omega) = -\delta(\underline{x} - \underline{x}')$$

Now no loss of generality to assume:

- spherical symmetry  $\Rightarrow$  propagation isotropic
- $\underline{x}' = 0$ .

$$\left( \nabla_r^2 + \frac{\omega^2}{c^2} \right) G = -\delta(r)$$

$$\text{now, } \nabla_r^2 G = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial G}{\partial r} \right)$$

$$= G'' + \frac{2}{r} \frac{\partial G}{\partial r}$$

$$\text{but } \frac{d^2}{dr^2} (rG) = \frac{d}{dr} (rG' + G)$$

$$= rG'' + 2G'$$

$$\frac{1}{r} \left( \frac{d^2}{dr^2} (rG) \right) + \left( \frac{(rG)\omega^2}{r c^2} \right) = -d(r)$$

$r d(r) = 0$ , except at  $r=0$

$$\frac{d^2}{dr^2} (rG) + \frac{\omega^2}{c^2} (rG) = 0$$

$$\Rightarrow G = \frac{A e^{i\frac{\omega}{c}r}}{r} + \frac{B e^{-i\frac{\omega}{c}r}}{r}$$

for  $r \ll 1$  reduction  $\Rightarrow A=B = 1/4\pi$

$$G(x-x', \omega) = \left( \frac{1}{4\pi|x-x'|} \right) \left[ e^{i\frac{\omega}{c}|x-x'|} + e^{-i\frac{\omega}{c}|x-x'|} \right]$$

$$G(x-x', t-t') = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G(\omega, x-x')$$

$$G = \frac{1}{4\pi|x-x'|} \left\{ \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega \left( \frac{|x-x'|}{c} - (t-t') \right)} \right.$$

$$+ \left. \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega \left( \frac{|x-x'|}{c} + (t-t') \right)} \right\} \quad (1)$$

$$= \frac{1}{4\pi|x-x'|} \left\{ \delta \left( \frac{|x-x'|}{c} - (t-t') \right) \right. \quad (2)$$

$$+ \left. \delta \left( \frac{|x-x'|}{c} + (t-t') \right) \right\}$$

Note G will sample source field at:  
 $\rightarrow$  time of flight from  $x' \rightarrow x$

$$t' = t - \frac{|x-x'|}{c} \rightarrow \text{retarded time (physical)}$$

$\Rightarrow$  outgoing wave

$$t' = t + \frac{|x-x'|}{c} \rightarrow \text{advanced time (future strikes present)}$$

unphysical!

reject term (2)

$\Rightarrow$  incoming wave.

∴ G.F. for Wave Equation is:

$$G = \frac{\delta\left(\frac{|x-x'|}{c_s} - (t-t')\right)}{4\pi|x-x'|}$$

b.) Contour Integration

→ Causality

i.e. consider:

→ localized disturbance at  $\underline{x} = 0, t = 0$

→ causality  $\Rightarrow G(\underline{x}, t) = 0, t < 0$ .

Now,

$$\tilde{G} = \int_{-\infty}^{+\infty} dt e^{i\omega t} G(\underline{x}, t) = \int_0^{\infty} dt e^{i\omega t} G(\underline{x}, t)$$

Further assume:

$$\int_{-\infty}^{\infty} G(\underline{x}, t) dt < \infty$$



∴  $\tilde{G}(x, \omega=0)$  exists and is bounded.

Now For general, complex values of  $\omega = \omega_1 + i\omega_2$

we have

$$\tilde{G}(x, \omega_1 + i\omega_2) = \int_0^{\infty} dt e^{i\omega_1 t} e^{-\omega_2 t} G(x, t)$$

Now

$$|\tilde{G}(x, \omega)| \leq \int_0^{\infty} dt e^{-\omega_2 t} |G(x, t)| \leq \int_0^{\infty} dt |G(x, t)|$$

}  
phase  $\Rightarrow$  oscillations

$< \infty$   
 $\downarrow$

{  $G(x, t)$  absolutely integrable

∴  $\tilde{G}(x, \omega)$  bounded in upper half plane ( $\omega_2 > 0$ )

also

all  $d^n \tilde{G} / d\omega^n$  exist

$\Rightarrow \tilde{G}(x, \omega)$  is analytic in UHP

→ How is this used?  $\Rightarrow$  Pole Prescription!

$$\nabla^2 G - \frac{1}{c^2} \partial_t^2 G = -\delta^3(\underline{x}) \delta(t)$$

$$\left( \frac{\omega^2}{c^2} - k^2 \right) \tilde{G}(\underline{k}, \omega) = -1$$

$$\Rightarrow \tilde{G}(\underline{k}, \omega) = \frac{1}{k^2 - \omega^2/c^2}$$

and understood  $\omega \rightarrow \omega + i\eta$   
 $\lim_{\eta \rightarrow 0}$

$$\tilde{G}(\underline{x}, \omega) = \int \frac{d^3k}{(2\pi)^3} e^{i\underline{k} \cdot \underline{r}} \tilde{G}(\underline{k}, \omega)$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty k^2 dk \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{e^{i\underline{k} \cdot \underline{r}}}{k^2 - \omega^2/c^2}$$

$$r = |\underline{x}|$$

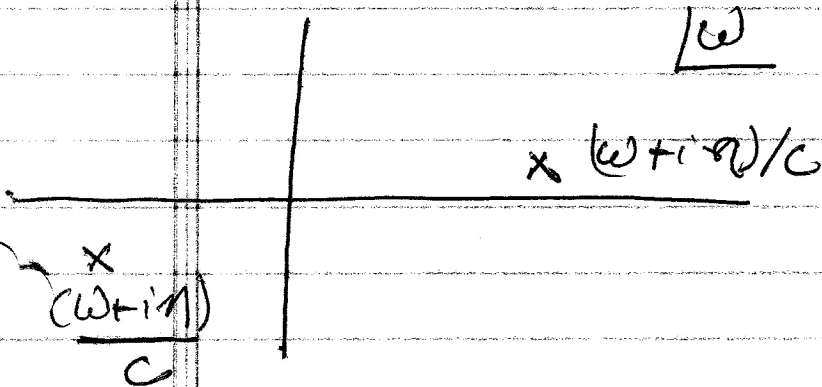
$$r = |\mathbf{x}|$$

Angular integrals  $\Rightarrow$

$$\tilde{G}(r, \omega) = \frac{1}{8\pi^2 i \epsilon} \int_{-\infty}^{\infty} k dk \frac{e^{i k r} - e^{-i k r}}{(k^2 - \omega^2/c^2)}$$

①                      ②

Now, for integral,  $\omega \rightarrow \omega + i\eta$



Now, - 2 terms  $\Rightarrow$  2 integrals + contour prescriptions  
 - 2 poles  $k = \pm \frac{\omega + i\eta}{c}$

$$\textcircled{1} \Rightarrow r \geq 0$$

$\Rightarrow \tilde{G}$  analytic

$$k = \frac{\omega + i\eta}{c} \quad ; \quad \text{c.e. } e^{i \frac{(\omega + i\eta)}{c} r} \sim e^{-\eta r/c} \rightarrow 0$$

converges

$$\textcircled{2} \rightarrow r > 0$$

$\rightarrow \tilde{G}$  analytic

$$\text{Kipole} = -\frac{(\omega + im)}{c}, \quad e^{+c(\frac{\omega + im}{c})r} \sim$$

$$e^{-\frac{\omega r}{c}} \quad \checkmark$$

$$\tilde{G}(\Omega, \omega) = \frac{1}{8\pi^2 r} \left[ \frac{\cancel{K} \cancel{2\pi} \cancel{c}}{\cancel{r} \omega} e^{i\frac{\omega}{c}r} \right.$$

$$\left. + \frac{\cancel{K} (+2\pi)}{\cancel{r} \omega} e^{i\frac{\omega}{c}r} \right]$$

$$= \frac{1}{4\pi r} e^{i\frac{\omega}{c}r}$$

so finally obtain:

$$\tilde{G}(\Omega, \omega) = \frac{1}{4\pi r} e^{i\frac{\omega}{c}r} \quad \checkmark$$

## (ii) Greens Functions + Boundary

→ major challenge of radiation theory :

reconciles  $\left\{ \begin{array}{l} \text{Green's function} \\ \text{boundary conditions} \end{array} \right.$

$$\text{A.B. : } G(\underline{x}-\underline{x}', t-t') = \frac{\delta\left(\frac{|\underline{x}-\underline{x}'|}{c} - (t-t')\right)}{4\pi|\underline{x}-\underline{x}'|}$$


is free space G.F. (retarded)

• can consider cases:

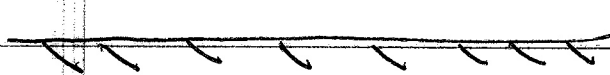
① homogeneous boundary condition

in homogeneous acoustic wave equation (source present)

i.e.

 monopole

pressure release



hard bndry or  
 $v_n = 0$

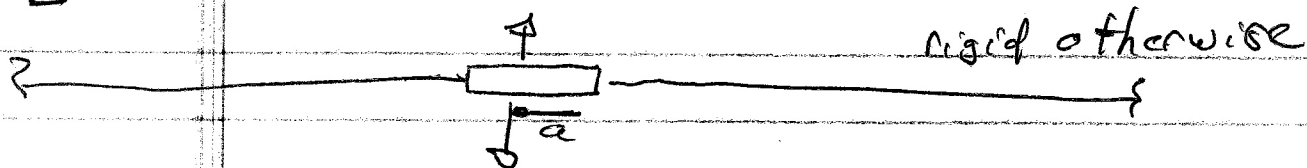
soft bndry  
 $p = 0$   
bndry

⇒ method of images!

[cf: 201, 203A, B]

② homogeneous equation (no source)  
+ inhomogeneous boundary

i.e.



circular piston:  $\delta h = \epsilon a e^{-i\omega t}$   
 $\downarrow$   
 $\ll 1$

i.e. piece of boundary pulsates

$\Rightarrow$  inhomogeneous boundary condition

$$\left. \frac{\partial \phi}{\partial z} \right|_0 = \begin{cases} 0, & r > a \quad (\text{rigid}) \\ v_n = -i\omega \epsilon a, & r < a \quad \text{piston motion} \end{cases}$$

Green's integral

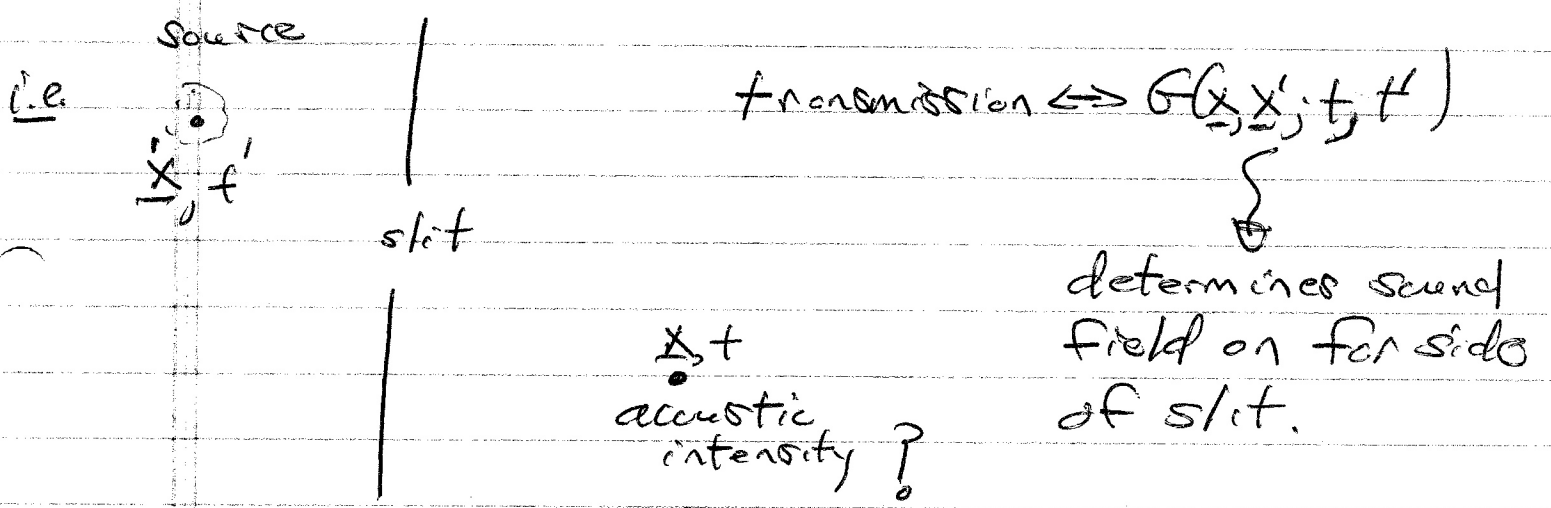
Physical Insight: Represent piston emission as emission of spherical waves associated with motion of points on piston head



etc.

⊕ Inhomogeneous boundary  
+  
Inhomogeneous Acoustic Egn. (Sources)

= Diffraction



- Cases:
- Fresnel
  - Fraunhofer

so

⇒ Case (b): Piston

First: Generalities

homogeneous boundary

b.c.

$$\alpha \frac{\partial \phi}{\partial z} - \beta \phi = 0 \quad | \quad z=0$$

$\beta=0 \rightarrow$  hard  
 $\alpha=0 \rightarrow$  soft

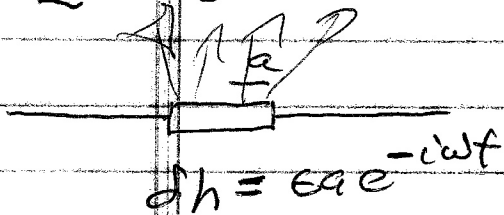
then, for G.F. for  $z > 0$ , apply 1D techniques to:

$$\frac{\partial^2 G}{\partial z^2} + \left( \frac{\omega^2}{c_s^2} - k_{\perp}^2 \right) G = -\delta(z-z')$$

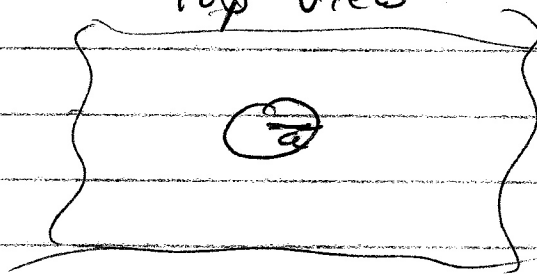
with b.c.

Now  $\rightarrow$  the Piston

side view



top view



b.c.'s on homogeneous:

$$\frac{\partial \phi}{\partial z} \Big|_0 = \begin{cases} 0, & r > a \\ -i\omega \epsilon a, & r < a \end{cases}$$

How deal with this } - Green's Identity }



Consider 2 fields:  $\phi, G$

1  $\rightarrow \phi$ : sound field

$$\nabla^2 \phi + \frac{\omega^2}{c^2} \phi = 0 \quad (1) \quad \leadsto \text{homog. field}$$

$$\hat{n} \cdot \nabla \phi = F(\underline{x}) \quad \leadsto \text{in homogeneous b.c.}$$

$\left\{ \begin{array}{l} \text{normal to} \\ z=0 \text{ plane} \end{array} \right.$

and

2  $\rightarrow G(\underline{x}, \underline{x}')$ : Green's function

$$\nabla^2 G + \frac{\omega^2}{c^2} G = + \delta(\underline{x} - \underline{x}') \quad (2)$$

with

$$\hat{n} \cdot \nabla G = 0 \quad \left\{ \begin{array}{l} \text{key point: } G \text{ is G.F.} \\ \text{for rigid, } \underline{\underline{\text{homog.}}} \text{ boundary} \\ \text{condition} \end{array} \right.$$

Now,  $\phi \otimes (2) - G \otimes (1) \Rightarrow$

$$\Delta = \phi \nabla^2 G + \frac{\omega^2}{c^2} \phi G + \phi \delta(\underline{x} - \underline{x}') - G \nabla^2 \phi - \frac{\omega^2}{c^2} G \phi$$

So, define  $I_G \equiv$  Green's Integral

$$I_G = \int d^3x' [\phi \nabla^2 G - G \nabla^2 \phi] = \int \phi(\underline{x}') \delta(\underline{x} - \underline{x}') d^3x$$

$$= \phi(\underline{x})$$

Further

(gives expression for sound field)

$$I_G = \int d^3x' [\phi \nabla \cdot \nabla G - G \nabla \cdot \nabla \phi]$$

$$= \int d^3x' (\nabla \phi \cdot \nabla G - \nabla G \cdot \nabla \phi) + \int da \cdot (\phi \nabla G - G \nabla \phi)$$

have:

$$\int da \cdot (\phi \nabla G - G \nabla \phi) = \phi(\underline{x})$$

$\int da$  is over  $z=0$  surface

Now,  $\hat{n} \cdot \nabla G = 0$   $\therefore da \cdot \nabla G = 0$

$$da \cdot \nabla \phi = da \hat{n} \cdot \nabla \phi = F$$

$$\hat{n} \cdot \nabla \phi|_{\text{surf}} = F$$

$$\boxed{+ \int da G(\underline{x}, \underline{x}') F(\underline{x}') = \phi(\underline{x})}$$

expression for  
sound field!

→ the Piston: G. For fixed  $\omega$

$$\begin{aligned}\phi(\underline{x}) &= + \int da \, G_{\omega}(\underline{x}, \underline{x}') F(\underline{x}') \\ &= + \int_{r < a} dx' dy' \, G_{\omega}(\underline{x}, \underline{x}') (-i\omega \epsilon a)\end{aligned}$$

$$G_{\omega}(\underline{x}, \underline{x}') = \frac{e^{i\frac{\omega}{c}|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|}$$

$$\phi(\underline{x}) = -i\omega \epsilon a \int_{r < a} dx' dy' \frac{e^{ik|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|}$$

now,  $|\underline{x}'| \sim a \ll |\underline{x}|$ , in far field

$$\begin{aligned}|\underline{x}-\underline{x}'| &= \left[ (\underline{x}-\underline{x}')^2 \right]^{1/2} \\ &= |\underline{x}| \left( 1 - 2\frac{\underline{x} \cdot \underline{x}'}{x^2} + \frac{x'^2}{x^2} \right)^{1/2}\end{aligned}$$

standard approx  
→ appears frequently

$$\approx |\underline{x}| \left( 1 - \frac{\underline{x} \cdot \underline{x}'}{x^2} \right) = |\underline{x}| - \hat{\underline{x}} \cdot \underline{x}'$$

$$k = \omega/c$$

$$\phi(\underline{x}) = \frac{-i\omega\epsilon_0 a}{4\pi} \int dx' \int dy' \frac{e^{ik|\underline{x}|}}{|\underline{x}|} e^{-ik\underline{\hat{x}} \cdot \underline{x}'}$$

$$\phi = -\frac{i\omega\epsilon_0 a}{4\pi} \frac{e^{ikr}}{r} \int dx' \int dy' e^{-ik\underline{\hat{x}} \cdot \underline{x}'}$$

Now, need perform I

$$I = \int dx' \int dy' e^{-ik\underline{\hat{x}} \cdot \underline{x}'}$$

$$\text{Now, } \int dx' \int dy' = \int_0^a dr' r' \int_0^{2\pi} d\phi'$$

ch. e.m.  $\rightarrow$   
expansion in  
 $kr' \ll 1 \rightarrow$   
magnetic dipole,  
quadropole  
terms

$$\underline{x}' = (r' \cos\phi', r' \sin\phi', 0)$$

$$\underline{\hat{x}} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$\underline{\hat{x}} \cdot \underline{x}' = r' \sin\theta (\cos\phi' \cos\phi + \sin\phi' \sin\phi)$$

$$= r' \sin\theta \cos(\phi' - \phi)$$

$$I = \int_0^a dr' r' \int_0^{2\pi} d\phi' e^{-ikr' \sin\theta \cos(\phi' - \phi)}$$

$$\cos(\phi' - \phi) = \sin(\phi' - \phi + \pi/2)$$

$$I = \int_0^a dr' r' \int_0^{2\pi} d\phi' e^{-ikr' \sin\theta \sin(\phi' - \phi + \pi/2)}$$

$$e^{ix \sin\phi} = \sum_n J_n(x) e^{in\phi} \quad (\text{standard identity})$$

$$\therefore I = \int_0^a dr' r' \int_0^{2\pi} d\phi' \sum_n J_n(-kr' \sin\theta) e^{in(\phi' - \phi - \pi/2)}$$

Kills all but  $n=0$

$$I = 2\pi \int_0^a dr' r' J_0(kr' \sin\theta)$$

$$\text{as } J_0(x) = J_0(-x)$$

Now, 
$$I = 2\pi \int_0^a dr' r' J_0(kr' \sin \theta)$$

Now, 
$$\frac{d}{dz} (z^n J_n(z)) = z^n J_{n-1}(z)$$

$\therefore \int_0^z J_0(\rho) \rho d\rho = z J_1(z)$

$\Rightarrow \phi(x) = \frac{-i\omega \epsilon_0 q}{4\pi r} e^{ikr} (2\pi a^2) \frac{J_1(ka \sin \theta)}{ka \sin \theta}$

$$= \pi a^2 v_p \frac{e^{ikr}}{r} \left[ \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right]$$

area velocity  
Piston

pt. response.

angular directivity factor

Note:  $\rightarrow ka \ll 1$  — low frequency limit

$$\lim_{x \rightarrow 0} \frac{2J_1(x)}{x} \cong 1 + O(x^2)$$

$$\phi(x) \cong \underbrace{\pi a^2}_{\text{area}} \underbrace{v_p}_{\text{velocity}} \frac{e^{ikr}}{r} \quad \text{dims } \checkmark$$

$\rightarrow$  pt. response.

→ For radiated power

$$\underline{S} = \epsilon_0^2 \overline{\underline{\tilde{p}} \underline{\tilde{v}}}$$

$$= \frac{1}{2} \epsilon_0 \underline{\tilde{v}} \underline{\tilde{p}}$$

$$\underline{\tilde{v}} = \underline{\nabla} \phi, \quad \underline{\tilde{p}} = - \frac{\partial \underline{\tilde{\phi}}}{\partial t}$$

1/6

$$\underline{S} = \frac{\rho_0 \epsilon^2 C_s^3 k^4 a^6}{8\pi^2} \left[ \frac{2J_1(k a \sin \theta)}{k a \sin \theta} \right]^2 \hat{r}$$

Radiated wave energy density flux

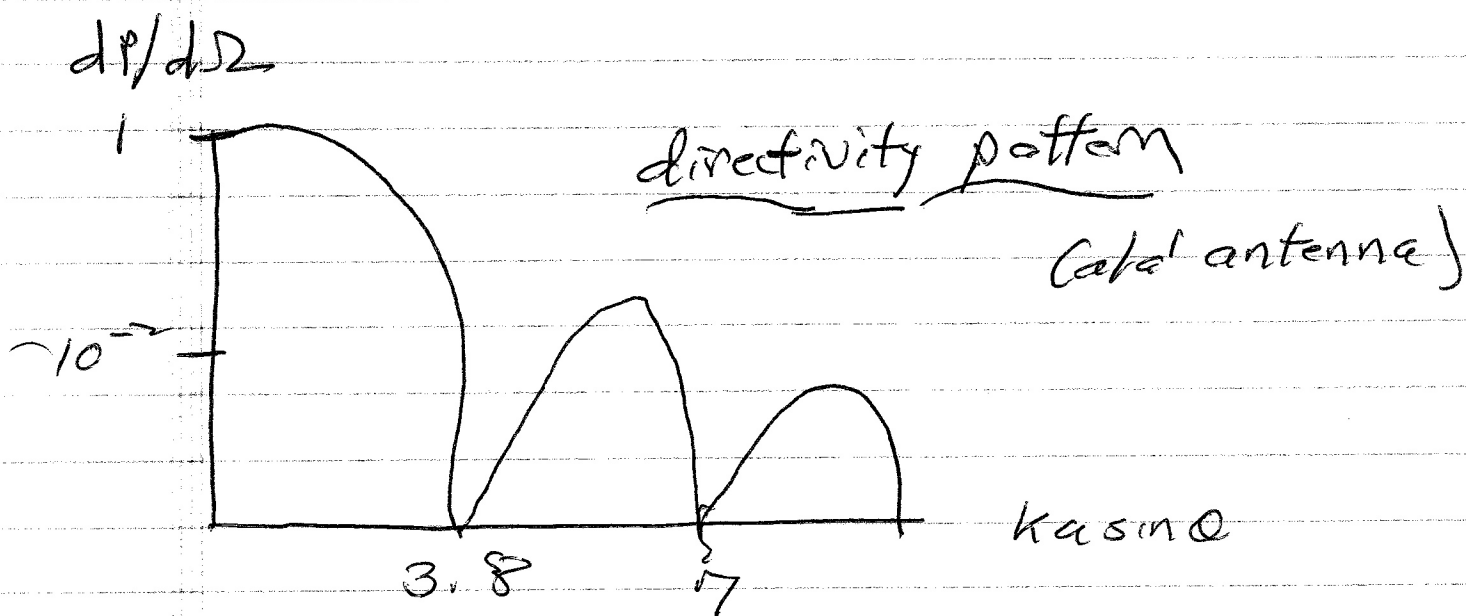
→ for  $dP/d\Omega$  → radiated power thru angular element  $d\Omega$

$$dP = \underline{S} \cdot \hat{r} dA$$

$$= \underline{S} \cdot \hat{r} r^2 d\Omega$$

$$\therefore \frac{dP}{d\Omega} = \frac{1}{8} \rho e^2 c_s^3 k^4 a^6 \left[ \frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right]^2$$

$\underbrace{\rho e^2 c_s^3 k^4 a^6}_{\epsilon_0^2 k^4 a^4}$ 
 $\underbrace{\left[ \frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right]^2}_{\text{directivity factor}}$



→ radiated power maximal forward (obvious)

→  $ka \ll 1 \Rightarrow$  isotropic radiation

→ for zero in radiated power

$ka \sin\theta = \alpha_{1,n} \rightarrow n^{\text{th}}$  zero of  $J_1$

$n=1 \rightarrow$  bndry of first lobe

$n=2 \rightarrow$  bndry of 2<sup>nd</sup> lobe



→ compare pulsating sphere;

$$\left. \begin{array}{l} kR \ll 1 \\ kR_0 \ll 1 \end{array} \right\}$$

$$P_{rad} = 2\pi\rho \omega^2 R^4 \frac{U_0^2}{c_s}$$

here

$$\frac{dP}{d\Omega} = \frac{1}{8\pi} \rho \frac{c_s^3}{c_s^4} \omega^4 a^6 \left[ \quad \right]^2$$

$$= \frac{1}{8\pi} \rho \frac{[c_s^2 a^2 \omega^2]}{c_s} \omega^2 a^4 \left[ \quad \right]^2$$

same as up to - numerical factor  
- directivity factor.

∴ not surprising as piston radiation clearly monopole.

→ Example 2: Oscillating Sphere

Familiar problem of oscillating sphere

$$a \circ \uparrow \downarrow \quad \vec{p} = \epsilon q e^{-i\omega t} \hat{z}$$

(dipole radiator) is also case of inhomogeneous b.c. + homogeneous source field.

Here, inhomogeneity due motion of sphere's surface!

Now:

- could proceed via Green's integral
- or
- undertake direct approach, using b.c.

Now, will go direct route:

$$\underline{z}_{sph} = \epsilon q e^{-i\omega t} \hat{z}$$

$$\underline{V}_{sph} = -i\omega \epsilon q e^{-i\omega t} \hat{z}$$

and have b.c.

$$V_r \Big|_a = \frac{\partial \phi}{\partial r} \Big|_a = -i\omega \epsilon q e^{-i\omega t} \hat{z} \cdot \hat{r}$$

for  $\phi_{\omega}$ ;

$$\left( \vec{z} \cdot \vec{r} = r \cos \theta \right)$$

$$\left. \frac{\partial \phi}{\partial r} \right|_a = -i\omega \epsilon_0 a \cos \theta$$

b.c

Now,  $\phi$  satisfies Helmholtz, so:

$$\nabla^2 \phi + \frac{\omega^2}{c^2} \phi = 0$$

in spherical geometry: Helmholtz Eqn.

$$\left( \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{L^2}{r^2} + \frac{\omega^2}{c^2} \right) \phi = 0$$

$$L^2 = -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

now, write:

spherical harmonics.

$$\phi(r, \theta, \phi) = R(r) Y_{lm}(\theta, \phi)$$

$$L^2 Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi)$$

(8)

have:

for  $R(r)$ ;

$$\left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{\omega^2}{c^2} \right] R = 0$$

spherical Bessel eqn.  $\rightarrow$  divergent,  $r \rightarrow 0$   
 (excluded in H-atom problem)  
 $R = (a j_l(kr) + b n_l(kr))$   
 $\left. \begin{array}{l} \text{finite, } r \rightarrow 0 \\ \text{finite, } r \rightarrow \infty \end{array} \right\}$

but, for radiation, seek outgoing wave :

$R = a e^{i k r}$   
 $\left. \begin{array}{l} \text{spherical Hankel fctn.} \\ \text{Hankel fctn.} \end{array} \right\} h_l = j_l + i n_l$   
 $\left. \begin{array}{l} h_l \sim e^{i k r} \\ h_l \sim e^{-i k r} \end{array} \right\}$

Now as  $\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = +i \omega a \cos \theta$

$$\phi(x) = C Y_{1,0}(\theta, \phi) f(kr) \quad (l=1)$$

$$= C \cos \theta f(kr)$$

( $r \neq 0 \rightarrow n$ )  
 allowed)

and, applying outgoing wave b.c. ;

$$f(kr) = h_1(kr) = - (kr + i) (kr)^{-2} e^{i kr}$$

$$\phi(r, \theta) = C \cos \theta f(kr)$$

$$f(kr) = h_1(kr)$$

finally,

$$C \cancel{\cos \theta} k f'(kr) = i \omega \epsilon_0 a \cancel{\cos \theta}$$

$$\Rightarrow C = \frac{i \omega \epsilon_0 a}{k f'(ka)} = \frac{i \epsilon_0 \epsilon a}{f'(ka)}$$

$$\phi(r, \theta, t) = \text{Re} \left( \frac{i \epsilon_0 \epsilon a \cos \theta f(kr) e^{-i \omega t}}{f'(ka)} \right)$$

→ complete solution.

→ clearly dipole ( $\sim \cos \theta$ )

Now,  $kr \gg 1 \iff$  far field

$$\phi(r, \theta, t) \approx -r_0 \left[ \frac{C \cos \theta e^{i(kr - \omega t)}}{kr} \right]$$

$$\Rightarrow V_r \approx -i C \cos \theta \frac{e^{i(kr - \omega t)}}{r}$$

$$\vec{p} \approx -i \epsilon_0 p_0 C \cos \theta \frac{e^{i(kr - \omega t)}}{r}$$

$$\begin{aligned} \therefore \frac{dP_{\text{rad}}}{d\Omega} &= \frac{r^3}{2} \operatorname{Re}(\tilde{p} \tilde{V}_r^*) \\ &= \frac{1}{2} c_s^2 |d|^2 \cos^2 \theta \end{aligned}$$

 $kr \gg 1$ 

finally, plugging in:

 $ka \text{ arb.}$ 

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{1}{2} c_s^3 (ka)^6 \frac{(c_s a)^2 \cos^2 \theta}{4 + k^4 a^4}$$

$$P_{\text{rad}} = \int d\Omega \frac{dP_{\text{rad}}}{d\Omega} = \frac{2\pi}{3} \left[ \frac{c_s^3 \rho_0 (c_s a)^2 (ka)^6}{4 + k^4 a^4} \right]$$

 $ka \ll 1$ 

$$P_{\text{rad}} \approx \frac{2\pi}{3} c_s^3 \rho_0 (a^6)^2 \frac{\omega^6 a^6}{c_s^6}$$

$$= \frac{2\pi}{3} \frac{\rho_0}{c_s^3} (\omega a^6)^2 \frac{\omega^4 a^6}{4}$$

$$= \frac{2\pi}{3} \frac{\rho_0}{c_s^3} |\tilde{u}_0|^2 \frac{\omega^4 a^6}{4} = \frac{\pi}{6} \frac{\rho_0}{c_s^3} |\tilde{u}_0|^2 \omega^4 a^6$$

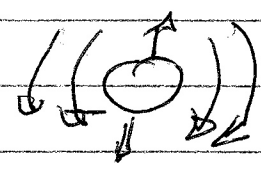
Recall, from Eq. 81;  $kr \gg 1$   
 $kr \ll 1$

$$P_{rad} = \frac{\pi}{6} \frac{\rho_0}{c_0^3} R^6 \omega^4 |U_0|^2 \quad \checkmark$$

check:

Aside:

- note here motion prescribed
- if external force applied to move sphere, would need account for induced mass

c.e.  flow of H<sub>2</sub>O about sphere.

$$F_{ext} = m_{sp} \frac{dy}{dt} + \frac{d}{dt} P_{H_2O}$$

Now is previous calculation of  $M_{ind}$  valid here?

No! → Radiation occurs ↔ outgoing wave b.c.  
⇒ energy propagates → ∞. Not closed system, as before!

How calculate induced mass?

→ Interpret induced mass as reaction force

c.e. sphere radiates, loses energy ↔ equivalent to reaction force acting on sphere,  
c.e.



$$F_{\text{ext}} = m_{\text{sph}} \frac{dy}{dt} + \frac{d}{dt} P_{\text{H}_2\text{O}}$$

$$F_{\text{ext}} - \frac{d}{dt} P_{\text{H}_2\text{O}} = m_{\text{sph}} \frac{dy}{dt}$$

$$P_{\text{H}_2\text{O}} = M_{\text{ind}} U_k$$

↑  
induced mass

can re-write:  $F_{\text{react}} = -\frac{d}{dt} P_{\text{H}_2\text{O}}$

$$F_{\text{ext}} + F_{\text{react}} = m_{\text{sph}} \frac{dy}{dt} \quad \frac{de}{dt} \quad \uparrow P_{\text{ext}}$$

$$F_{\text{react}} = \int_{\text{sphere}} d\mathbf{A} \cdot \hat{\mathbf{r}} \tilde{\rho}(\theta, \phi) \Big|_{r=a}$$

here, need  $(F_{\text{react}})_z \Rightarrow$

$$F_z = \int d\mathbf{A} \cdot \hat{\mathbf{z}} \tilde{\rho}(\theta, \phi) \Big|_{r=a}$$

$$-d\mathbf{A} = \hat{\mathbf{r}} dA$$

$$\begin{aligned} \vec{p} &= -\rho_0 \frac{\partial \phi}{\partial t} \\ &= -\rho_0 \omega_0 a \epsilon \cos \theta \frac{f(kr)}{f'(ka)} \end{aligned}$$

from  $\vec{z} \cdot \vec{r}$  and  $\vec{p}$ 

$$F_z^{\text{react}} = -a^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \left( -\rho_0 \omega_0 a \epsilon \cos \theta \frac{f(kr)}{f'(ka)} \right)$$

$$= -\frac{4\pi}{3} \omega^2 a \epsilon \rho_0 a^3 \frac{f(ka)}{(ka) f'(ka)}$$

$$= \underbrace{\omega^2 a \epsilon (\rho_0 V)} \left[ \frac{2 + k^2 a^2 + ik^3 a^3}{4 + k^4 a^4} \right]$$

$$V = \frac{4\pi a^3}{3}$$

$$\rho_0 V = M_{\text{fluid displ.}}$$

$$\begin{aligned} \text{Now, } F_{\text{ext } z}(\omega) + F_{\text{react } z}(\omega) &= M_{\text{sph}} \frac{d^2 z_{\text{sph}}}{dt^2} \\ &= -M_{\text{sph}} \omega^2 \epsilon a \end{aligned}$$

So can re-write:

$$\frac{F_{ext}}{2}(\omega) = -\omega^2 \epsilon a \left[ M_{sph} + \rho_0 V \left[ \frac{2 + k a^2 + i k a^3}{4 + k a^4} \right] \right]$$

induced mass

$$M^* = M_{eff}$$

net effective mass

→ for  $ka \rightarrow 0 \iff$  must recover potential flow

$$ka \rightarrow 0 \quad M_{ind} = \rho_0 V \frac{2}{4} = \frac{\rho_0 V}{2} \quad \checkmark$$

→ finite  $ka \uparrow \Rightarrow M_{ind}$  is complex  $\uparrow \uparrow$

observe if damping:

$$F_{ext} - \eta \dot{y} = m \frac{dy}{dt}$$

$$F_{ext} = -\omega^2 m \tilde{z} + i \omega \eta \tilde{z}$$

⊗ complex inertia!

imaginary piece  $M_{\text{rad}} \leftrightarrow$  damping

origin of damping  $\rightarrow$  radiation of energy to  $\infty$

$\rightarrow$  outgoing wave b.c.'s